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Stress-deformed state of cylindrical specimens during indirect tensile strength testing

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ABSTRACT

In this study, the interaction between cylindrical specimen made of homogeneous, isotropic, and linearly elastic material and loading jaws of any curvature is considered in the Brazilian test. It is assumed that the specimen is diametrically compressed by elliptic normal contact stresses. The frictional contact stresses between the specimen and platens are neglected. The analytical solution starts from the contact problem of the loading jaws of any curvature and cylindrical specimen. The contact width, corresponding loading angle (\(2\theta_0\)), and elliptical stresses obtained through solution of the contact problems are used as boundary conditions for a cylindrical specimen. The problem of the theory of elasticity for a cylinder is solved using Muskhelishvili’s method. In this method, the displacements and stresses are represented in terms of two analytical functions of a complex variable. In the main approaches, the nonlinear interaction between the loading bearing blocks and the specimen as well as the curvature of their surfaces and the elastic parameters of their materials are taken into account. Numerical examples are solved using MATLAB to demonstrate the influence of deformability, curvature of the specimen and platens on the distribution of the normal contact stresses as well as on the tensile and compressive stresses acting across the loaded diameter. Derived equations also allow calculating the modulus of elasticity, total deformation modulus and creep parameters of the specimen material based on the experimental data of radial contraction of the specimen.

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1. Introduction

The indirect tensile testing method, known as the Brazilian test, provides an alternative to direct tensile testing of cylindrical rock specimens and other brittle materials clamped between two loading flats or arcs. The International Society for Rock Mechanics (ISRM) (1978) officially suggested the Brazilian test as a method for determining the tensile strength of rock materials. The standard test method can be followed according to ASTM D3967-08 (2008). The European standard for testing the tensile strength of concrete specimens was approved by European Committee for Standardization (CEN) on 18 February 2000 (EN 12390-6, 2000). The history of development and applications of the Brazilian test in rock mechanics has been reviewed and investigated most recently by numerous scientists, e.g. Daemen et al. (2006), Ye et al. (2009), and Li and Wong (2013).

Carneiro and Barcellos (1953), the pioneers of the Brazilian test method, used Hertz (1889) analysis of the action of concentrated compressive forces \(P\) in diametrically opposite points of a cylindrical disk with radius \(R_1\) and length \(L\) (Fig. 1a) as the theoretical basis of the method. According to this analytical model, the principal tensile stresses \(\sigma_x\) along the vertical diameter are uniformly distributed and given by

\[
\sigma_x = \frac{P}{\pi R_1 L}
\]

(1)

The normal principal compressive stress \(\sigma_y\) is given by

\[
\sigma_y = -\frac{P}{\pi R_1 L} \frac{3R_1^2 + r^2}{R_1^2 - r^2}
\]

(2)

where \(r\) is the distance from the origin, and \(-R_1 < r < R_1\). In the center (\(r = 0\)), \(\sigma_y = -3P/(\pi R_1 L)\), and it is infinite on the periphery points (\(r = \pm R_1\)) of the disk.

The uniform distribution of tensile stresses (Eq. (1)) as well as the preparation of cylindrical specimens from core material with ease stimulate the wide application of the Brazilian test method as standard for brittle materials. Such an approach can have an adequate accuracy and the error is small compared to direct tensile testing, when the experimental installation provides the action of
almost concentrated (abstract) forces \( P \) (Fig. 1a). In practice, this model could be approximated to reality in case by using sharp wedges or small-diameter steel rods (Fig. 1b) between the loading plates and the specimen. But in such a case the line load creates extremely high contact stresses, causing different modes of failure if destruction of the specimen starts at the contact points due to highly intensive shear stresses, rather than in the center because of tension. Therefore, it is assumed that a wider contact strip can reduce these problems significantly and that an arc of contact smaller than 15\(^\circ\) causes no more than 2\% error in the principal tensile stress while reducing greatly the incidence of premature cracking (ASTM D3967-08, 2008).

When the specimen is placed directly between flat platens, or between indenters with nonzero radius of curvature of contacting surfaces, the application of Eq. (1) can cause a certain error, the magnitude of which should be estimated. A number of scholars have paid attention to this problem shortly after the popularization of the Brazilian test method, e.g. Hondros (1959), Jaeger and Cook (1964), Wijk (1978), Amadei (1983), Amadei et al. (1983), Chen et al. (1998), Lavrov and Vervoort (2002), Ye et al. (2009), Marion and Johnstone (1977), Procopio et al. (2003), Markides et al. (2010, 2011), Markides and Kourkoulis (2012), and Li and Wong (2013) suggested different analytical and numerical solutions and improved schemes, generalized for different kinds of anisotropy and homogeneity of testing rocks, concretes, glass, and many other brittle and not quite brittle materials (e.g. nuclear wastes (ASTM C1144-89, 1989), asphalt concrete).

Jaeger and Cook (1964) and Jaeger et al. (2007) have given an analytical approach for estimating the stress-strain state of the cylindrical specimen of elastic, isotropic rock. The distribution of tractions (Fig. 2) on the specimen surfaces corresponding to some central loading angle \( 2\theta_0 \) is uniform (Fig. 1b). The radial, tangential normal and shear stress components can be expressed respectively in the polar coordinates as

\[
\tau_{rr} = \frac{2P\theta_0}{\pi} - \frac{2P}{\pi} \sum_{m=1}^{+\infty} \left( \frac{r}{a} \right)^2 \left( 1 - \frac{1}{m}\right) \sin(2m\theta_0)\cos(2m\theta) \tag{3}
\]

\[
\tau_{\theta\theta} = \frac{2P\theta_0}{\pi} - \frac{2P}{\pi} \sum_{m=1}^{+\infty} \left( \frac{r}{a} \right)^2 \left( 1 + \frac{1}{m}\right) \sin(2m\theta_0)\cos(2m\theta) \tag{4}
\]

Along the \( x \) axis, where \( \theta = 0 \), \( \tau_{rr}(\theta = 0) = 0 \), \( -a \leq r \leq a \), and \( \rho = r/a \), the series in Eqs. (3) and (4) can be summed in the closed form (Hondros, 1959) to give

\[
\tau_{rr}(\theta = 0) = \frac{2P}{\pi} \left[ \frac{(1 - \rho^2)\sin(2\theta_0)}{1 - 2\rho^2\cos(2\theta_0) + \rho^4} + \arctan\left( \frac{1 + \rho^2}{1 - \rho^2} \tan\theta_0 \right) \right] \tag{5}
\]

\[
\tau_{\theta\theta}(\theta = 0) = -\frac{2P}{\pi} \left[ \frac{(1 - \rho^2)\sin(2\theta_0)}{1 - 2\rho^2\cos(2\theta_0) + \rho^4} - \arctan\left( \frac{1 + \rho^2}{1 - \rho^2} \tan\theta_0 \right) \right] \tag{6}
\]

Amadei (1983) and Chen et al. (1998) developed an analytical approach and computer program to determine the stress at any arbitrary point in the disc made of a transversely isotropic medium under diametrical loading uniformly distributed over a strip of the same contact angle \( (2\theta_0 = 15^\circ) \). It should be noted that such uniform distribution of contact stresses may be most characteristic of pretreated flattened cylindrical specimens (Fig. 1d) (e.g. Wang et al., 2004; Dave et al., 2011).

At present, most researchers usually assume a uniform distribution of contact stresses when using Brazilian tests. The uniform distribution of contact stresses seems highly unlikely. But many authors, e.g. Fairhurst (1964), Colback (1967), Vardar and Finnie (1975), Dan et al. (2013), have adopted this assumption. They argued that the details of the distribution of contact stresses should not be particularly relevant, given that the prime interest focuses on the characteristics of the rock failure at the center of the disk, i.e. far away from the applied (boundary) stresses.

As is noticed by Daemen et al. (2006), the distribution of contact stresses between the specimen and loading platens in a Brazilian test is a typical contact problem, which remains difficult to be modeled numerically (e.g. Hills et al., 1993). Since it is impossible for the real test conditions to meet all assumptions made for the theoretical development, they tried to determine the distribution of contact stresses experimentally using the pressure film of SPI Corporation.

Similar opinions are stated by Ye et al. (2009), who remarked that there is no way to know exactly the specific variable in the processing of loading of the contact angle \( 2\theta_0 \) of the stress distribution. The value of \( 2\theta_0 \) varies despite using the same concave
loading plates and under the same loading, due to the type of rock. These factors make it very difficult to calculate the contact angle. Accordingly, it is also very difficult to calculate exactly the stress field in the disc subjected to distributed loads over an arc when the disc is in the elastic stage.

There is recent work by Kourkoulis et al. (2012, 2013), Markides and Kourkoulis (2012) and others. They conducted intensive research on the problems of stress-deformed state of cylindrical specimens under parabolic contact loading employing the complex potentials method (Muskhelishvili, 1963). The author also paid attention to this problem, and some problems of stress-deformed state of cylindrical specimens under elliptic (parabolic) contact loading were investigated using the complex potentials method (Japaridze, 1971, 1972).

Thus, the problem of the shape of the distribution of normal and tangential contact tractions for the standardized indirect tensile splitting (“Brazilian”) test, and its influence on the stress concentration, tensile strength and other mechanical parameters of the rocks and other hard materials still remain topical.

The goal of this paper is to present some new insights into the Brazilian test. It provides a solution of a nonlinear interaction between the flat and curved loading platens and cylindrical specimens in the Brazilian test. The contact width, corresponding contact angle and elliptical contact stresses obtained from the solution of the contact problems are used as boundary conditions for the cylindrical specimen. The problem of the theory of elasticity for a cylindrical specimen is solved using the complex potentials method developed by Muskhelishvili (1963).

Numerical examples are shown to determine the influence of deformability and curvatures of specimen and bearing platens on the normal contact stresses, tensile and compressive stress concentration factors across the loaded diameter.

Suggested equations also allow calculating the modulus of elasticity, the total deformation modulus and the creep parameters of the specimen material on the basis of experimental data of radial contraction of the specimen.

2. Theoretical background

A cylindrical specimen with radius \( R_1 \) and length \( L \), and the loading jaws with radius of the contact faces \( R_2 \) compressed (Fig. 3) by forces \( P \), touch each other on surfaces with width \( 2a \).

According to the developments of the contact problem of the theory of elasticity (Hertz, 1895; Shtaerman, 1949; Muskhelishvili, 1963; Timoshenko and Goodier, 1970), the half width of the contact surface, \( a \), is given by

\[
a = \frac{4P}{\pi L \left[ f_1''(0) + f_2''(0) \right]} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \quad (8)
\]

where \( \nu_1, \nu_2 \) and \( E_1, E_2 \) are the Poisson’s ratios and moduli of elasticity of specimen and loading jaws, respectively; \( f_1''(0) \) and \( f_2''(0) \) are the second derivatives of the functions describing the surfaces of a compressible bodies at the point of initial contact. For the case of cylindrical surfaces, this equation takes the following form:

\[
a = \frac{4PR_1}{\pi L \left( 1 + R_1/R_2 \right)} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \quad (9)
\]

Consequently, the half loading angle can be written as

\[
\theta = \frac{1}{2} \sin^{-1} \left( \frac{a}{R_1} \right)
\]

Fig. 2. (a) Diametrical compression over two symmetrical arcs of a cylinder with radius \( a \), for the case of \( 2\theta = 15^\circ \); normalized against \( P \) (Jaeger et al., 2007).

Fig. 3. Compression of a cylindrical specimen made of homogeneous, isotropic and elastic material between two elastic jaws. \( p_{\text{max}} \) is the maximum contact stress; \( z_1 \) and \( z_2 \) are the boundary points of the contact surface.
where

\[ \theta_0 = \frac{\pi}{2} - \theta_1 = \arcsin\left(\frac{a}{R_1}\right) \]  

(10)

Eq. (9) relates the case of plane strain that would occur if \( L > R_1 \). When \( L < R_1 \), the thin cylindrical specimen will be closer to the "plane state stress" and Eq. (9) will be more correct if \( r_1=r_2=0 \) is used.

When considering the joint deformation of the contacting bodies, the components of displacements of their surface points may be fully identical ("no slip") and partially identical ("full slip") in relation to limiting frictional and/or cohesion contact stresses. In the first case, there will exist normal and tangential contact stresses, which may be interesting as an example in the rock-support interaction problems. In the second case, when friction and cohesive forces could not provide fully identical joint deformation, tangential contact stresses may be ignored.

Regarding the Brazilian test, the influence of dry friction stresses on disc-jaw contact surfaces is doubtful, because of several causes, such as small contact width and possibility of the relaxation of the surface friction stresses. At least, the use of various types of soft loading pads (ASTM C1144-89, 1989), bearing strips (ASTM C496/C496M-11, 2004) and others, practically exclude the appearance of significant tangential surface stresses. Thus the consideration of tangential contact stresses at the disc-jaw interaction for the more exact boundary conditions will cause an unnecessary complication of the analytical study of the Brazilian test.

If the tangential stresses are neglected (Shtaerman, 1949), the elliptic function of the normal contact stresses in polar coordinates is

\[ p(\theta) = \frac{2P}{\pi a} \sqrt{a^2 - R_1^2 \cos^2 \theta} \]  

(11)

where

\[ \frac{\pi}{2} - \theta_0 \leq \theta \leq \frac{\pi}{2} + \theta_0, \quad \frac{3\pi}{2} - \theta_0 \leq \theta \leq \frac{3\pi}{2} + \theta_0 \]  

(12)

Eq. (11) does represent an ellipse with small semi-axis as a half-width of contact surface (Eq. (9)) and large semi-axis \( 2P/(\pi a L) \) presenting the maximum contact stress \( p_{\text{max}} \) at the points \( \theta = \pi/2 \) and \( \theta = 3\pi/2 \), respectively.

Using series expansion, Eq. (11) with satisfactory accuracy for technical purposes can be expressed in the complex formulation as follows:

\[ p(\theta) = \frac{4P}{\pi L(z_1 + z_1)} \left[ 1 - \frac{R_1^2}{(z_1 + z_1)^2} - \frac{z^2 + z^2}{2(z_1 + z_1)^2} \right] \]  

(13)

where

\[ z = ra, \quad z_1 = R_1 e^{i(\pi/2 - \theta_1)}, \]  

(14)

\[ z_1 = R_1 e^{-i(\pi/2 - \theta_1)} \quad (0 \leq r \leq R_1) \]

The following notations will be used:

\[ A = \frac{4P}{\pi L(z_1 + z_1)} \left[ 1 - \frac{R_1^2}{(z_1 + z_1)^2} \right] \]  

(15)

\[ B = \frac{2PR_1}{\pi L(z_1 + z_1)^2} \]  

(16)

The points \( z \) and \( z_1 \) on the plane \( \zeta = z/r \) correspond to the points \( \sigma = e^{i\theta} \) and \( \sigma_1 = e^{i(\pi/2 - \theta_1)} \). Consequently, the boundary conditions can be written as

\[ \Phi(\sigma) + \Phi(\sigma) - \sigma \Phi'(\sigma) - \sigma^2 \Psi(\sigma) = A + B \left( \sigma^2 + \sigma^2 \right) \]  

(17)

Corresponding analytical functions of a complex variable, \( \Phi(\zeta) \) and \( \Psi(\zeta) \), which are holomorphic inside the contour \( \gamma \) of the circle for points \( \zeta = r \sigma \), are given by

\[ \Phi(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{N(\sigma)}{\sigma - \zeta} d\sigma - \frac{1}{4\pi i} \int_{\gamma} \frac{N(\sigma)}{\sigma} d\sigma \]  

(18)

\[ \Psi(\sigma) = -\frac{1}{2\pi i} \int_{\gamma} \frac{N(\sigma)}{\sigma(\sigma - \zeta)^2} d\sigma - \frac{1}{2\pi i} \int_{\gamma} \frac{N(\sigma)}{\sigma^2 (\sigma - \zeta)} d\sigma \]  

(19)

Returning to the previous variable \( z = \zeta R_1 \) from these equations yields:

\[ \Phi(z) = \frac{1}{2\pi i} \left[ \left( A + \frac{B R_1^2}{R_1^2} + \frac{BR_1^2}{z^2} \right) \ln \left( \frac{z_1^2}{z_1^2} \right) - \left( A + \frac{BR_1^2}{R_1^2} \right) \ln \left( \frac{z_1^2}{z_1^2} \right) \right] \]  

(20)

\[ \Psi(z) = \frac{1}{2\pi} \left[ \left( A + \frac{B R_1^2}{R_1^2} + \frac{BR_1^2}{z^2} \right) \left( \frac{R_1^2}{z_1^2} - \frac{R_1^2}{z_1^2} \right) - \left( A - \frac{BR_1^2}{R_1^2} \right) \ln \left( \frac{z_1^2}{z_1^2} \right) \right] \]  

(21)

The combination of stress components at any point of the disk can be found by substituting these functions into the well-known equations of Muskheilishvili (1963):

\[ \sigma_x + \sigma_y = 2 \left( \Phi(z) + \Phi(z) \right) \]  

(22)

\[ \sigma_y - \sigma_x + 2i \tau_{\varphi} = 2 \left[ \Phi(z) + \Psi(z) \right] \]  

(23)

The equations of the internal stresses at points on the \( y \) axis can be written in the analog form as Eqs. (1) and (2):

\[ \sigma_x(r) = K \frac{P}{R_1^2} \]  

(24)

\[ \sigma_y(r) = C \frac{P}{R_1^2} \]  

(25)

where \( K \) and \( C \) are the normalized concentration factors for tensile and compressive stresses, respectively (Chen et al., 1998), and can be expressed as
It is also interesting to determine an explicit measure of the rock stiffness, e.g.
in terms of its Young's modulus, on the basis of the simplest possible indirect tensile testing (e.g. Chen et al., 1998; Ye et al., 2009). If such a determination could be made reliably and consistently, it would provide an ideal tool to investigate the spatial variability of the rock stiffness, because it requires far less time for sampling, specimen preparation, testing, and data analysis than compressive testing (Daemen et al., 2006).

The radial and tangential displacement components $u$ and $v$ at any point of the disk can be found using the analytic complex variable functions, $\Phi(z)$ and $\Psi(z)$ by the well-known equations of Muskhelishvili (1963):

$$E_1 = \frac{3R_1^3}{20a^2} \left[ G \left( \frac{8a^2 - 3R_1^2}{R_1^2} + \frac{r^2}{R_1^2} \right) + H \left( \frac{R_1^4 - 4a^2}{R_1^2 - r^2} - \frac{R_1^4}{r^2} \right) - I \right]$$

(26)

$$C = \frac{3R_1^3}{20a^2} \left[ G \left( \frac{R_1^2 - 8a^2}{R_1^2 - r^2} + \frac{2r^2}{R_1^2} \right) - H \left( \frac{R_1^4 - 4a^2}{R_1^2 - r^2} - \frac{2R_1^4}{r^2} \right) - I \right]$$

(27)

where

$$G = \arctan \left( \frac{\sqrt{R_1^2 - a^2} - r}{a} \right) + \arctan \left( \frac{\sqrt{R_1^2 - a^2} + r}{a} \right) - \pi$$

(28)

$$H = 2\arccos \left( \frac{a}{R_1} \right) - \pi$$

(29)

$$I = \frac{2a(R_1^2 - r^2)\sqrt{R_1^2 - a^2}}{(R_1^2 - r^2)^2 + 4a^2r^2} \left( \frac{2R_1^2 - 8a^2}{R_1^2} - \frac{r^2}{R_1^2} \right)$$

(30)

In the particular theoretical case of absolutely rigid platens and disk ($E_1 = E_2 \to \infty$), or sharp indenters ($R_2 = 0$), according to Eqs. (3)–(5), $a \to 0$ and Eqs. (24) and (25) converge to the Hertz equations, i.e. Eqs. (1) and (2).

The components of displacements at any point of the cylindrical specimen may be determined by inserting the integrals of analytical functions (Eqs. (20) and (21)) into Eq. (24) and separating the real and imaginary parts of the resulting expression.

The analysis of these functions can be easily made using the computer program MATLAB. Results of calculations are given graphically in Fig. 4.

Fig. 4 shows that the magnitudes of tensile and compressive stresses in the internal part of the diameter of high-modulus specimens and flat loading platens are close to each other, whereas in the external areas the difference between the magnitudes obtained from Hordons’ solutions and the proposed analytical solution increases significantly. This difference is much larger in the external areas as well as in the internal part and even in the center of the disk, especially when the curved loading platens are used. This confirms the comment in the ISRM suggested methods (Brown, 1981) that a critical dimension of the apparatus is the radius of curvature of the jaws.

To determine the influence of elastic and geometric parameters on the loading angle and the stress concentration factors $K$ and $C$ acting across the loaded diameter, let cylindrical specimens with radius $R_1 = 5$ cm, length $L = 2.5$ cm, Young’s modulus $E_1 = 500$ MPa, $1000$ MPa, $2500$ MPa, $5000$ MPa, $15,000$ MPa, and Poisson’s ratio $\nu = 0.3$ be compressed diagnostically by force $P = 40$ kN.

The specimen is compressed between planar ($R_2 \to \infty$) steel ($E_2 = 200$ GPa) platens without pads (Fig. 1a). Appropriate values of the half width of contact surface, $a$, loading angle, $\theta_{\text{max}}$, maximum contact stress, $p_{\text{max}}$, contraction of radius along the $y$ axis, $\Delta R_1$, are respectively calculated by Eqs. (9)–(11) and (33), and the stress concentration factors (Eqs. (26) and (27)) are given in Table 1.
If the specimen is compressed between (R₂ = 8 cm) steel (E₂ = 200 GPa) platens (Fig. 1e), appropriate values of half width of contact surface, a, contact angle, 2θ₀, maximum of elliptical contact stresses, pₘₐₓ, contraction of radius, ΔR₀, and the stress concentration factors (Eqs. (26) and (27)) of specimen and the curvature of the bearing platens are larger. Both factors have less influence on the characteristics and magnitudes of the compressive stress and stress concentration factor, C, as shown in Fig. 6.

The relative distance r₁/R₁ for the initial data is given in Table 4. In the internal diametrical part of the disk (r₁ < r < R₁), negative stress, i.e. tensile normal stress, acts, and in the external parts r₁ < r < R₂, positive stress, i.e. compressive stress, acts.

The normalized vector sum F of the stresses acting on all three parts (Fig. 7) of a disk diameter (R₁ < r < R₂) was obtained from integration by parts of Eq. (24):

\[ F = F_{\text{tens}} + 2F_{\text{comp}} = \int_{r_1}^{r_2} \sigma_x dr + 2 \int \sigma_x dr \]  

where \( F_{\text{tens}} \) and \( F_{\text{comp}} \) are the normalized vector sums of the tensile and compressive stresses, respectively.

The magnitudes of the normalized vector sum F depend on Young’s modulus and type of loading platens calculated using MATLAB are also listed in Table 4. The results of specific numerical

| Table 2 | Numerical results of a half width of contact surface, a, contact angle, 2θ₀, maximum contact stresses, pₘₐₓ, contraction of radius, ΔR₀, along the y axis, and stress concentration factors K and C in the center (r = 0) and on the contour (r = R) of specimen for given values of applied load, P, geometric and elastic parameters of a specimen and curved loading platens (Fig. 1e) in function of Young’s modulus E₁ of a specimen (P = 40 kN, R₁ = 5 cm, R₂ = 8 cm, L = 2.5 cm, r = 0.3).
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<td>2θ₀ (°)</td>
<td>pₘₐₓ (MPa)</td>
<td>ΔR₀ (cm)</td>
<td>K (r = 0)</td>
<td>K (r = R)</td>
<td>C (r = 0)</td>
<td>C (r = R)</td>
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<td>-0.99</td>
<td>22.39</td>
<td>3.16</td>
<td>22.39</td>
</tr>
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</table>

| Table 3 | Influence of Poisson’s ratio of the specimen on the loading angle and stress concentration factors (E₁ = 1 GPa, P = 40 kN, R₁ = 5 cm, R₂ = 8 cm).
<table>
<thead>
<tr>
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<td>a (cm)</td>
<td>2θ₀ (°)</td>
<td>pₘₐₓ (MPa)</td>
<td>ΔR₀ (cm)</td>
<td>K (r = 0)</td>
<td>K (r = R)</td>
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</tr>
<tr>
<td>0.2</td>
<td>1.62</td>
<td>37.1</td>
<td>-0.894</td>
<td>5.24</td>
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<td>0.3</td>
<td>1.58</td>
<td>36.2</td>
<td>-0.89</td>
<td>5.36</td>
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<td>1.51</td>
<td>34.6</td>
<td>-0.908</td>
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<td>1.43</td>
<td>29.3</td>
<td>-0.919</td>
<td>5.84</td>
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</table>

Fig. 4. Comparison of stress concentration factors K and C along the compressed diameter (R₁ ≤ r ≤ R) from the developed analytical modeling results and Hondros’ solutions for (a) flat (R₂=+∞, E₁ = 20 GPa, 2θ = 15°) and (b) curved (R₂ = 8 cm, E₁ = 20 GPa) bearing platens.
examples show that the vector sum of the normal stresses acting on the whole disk diameter at the elastic stage are positive, i.e. compressive. This is because, on the two relatively small external parts \( r_1/C_2 C R \), the magnitude of the compressive forces \( 2F_{\text{comp}} \) is greater than the sum of the tensile stress \( F_{\text{tens}} \) on the internal \((r_1 < r < r_1)\) diametrical part.

The action of the high intensity compressive stresses in the external areas of the diameter may somewhat impede the spreading of the cracks from the central part of a disk due to tensile stresses. At the same time, the development of the oncoming cracks due to shear stresses in the peripheral parts can complete the splitting of disk. These cases have been observed in experiments and are described in numerous publications, more recently by Ye et al. (2009), Li and Wong (2013) and others. Such “tensile-shear” kind of the failure mode will cause a decrease of the indirect tensile test accuracy.

When the fracture deviates from the center too much due to shear stress effects, for example, in most of the more than a hundred experiments, Daemen et al. (2006) considered other modes of failure as invalid for determination of tensile strength. This should be considered to improve the modes of failure of specimens in the Brazilian test.

4. Conclusions

The analytical solutions of the plane stress-strain state problem are developed for the elastic, homogeneous, isotropic cylindrical specimen compressed between jaws of any curvature. The contact width, corresponding contact angle, \( 2\theta_0 \), elliptical contact stresses.

Table 4

<table>
<thead>
<tr>
<th>( E_1 ) (MPa)</th>
<th>( R_2 \rightarrow \infty )</th>
<th>( r_1/R_1 )</th>
<th>( F )</th>
<th>( r_1/R_1 )</th>
<th>( F )</th>
</tr>
</thead>
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<tr>
<td>500</td>
<td>0.66</td>
<td>0.94</td>
<td>0.61</td>
<td>1.64</td>
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<tr>
<td>1000</td>
<td>0.78</td>
<td>0.61</td>
<td>0.71</td>
<td>1.11</td>
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<td>2500</td>
<td>0.83</td>
<td>0.3</td>
<td>0.77</td>
<td>0.64</td>
<td></td>
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<tr>
<td>5000</td>
<td>0.87</td>
<td>0.11</td>
<td>0.82</td>
<td>0.39</td>
<td></td>
</tr>
<tr>
<td>15,000</td>
<td>0.9</td>
<td>-0.51</td>
<td>0.87</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>
obtained from the solution of contact problems from the theory of elasticity are used as boundary conditions for cylindrical specimens. The problem is solved using the method developed by Muskhelishvili (1963) according to which the stresses and displacements are represented in terms of analytic functions of a complex variable.

The resulting relations allow estimating the stress state in the specimens and the tensile strength of the cylindrical specimens of rock, concrete and other brittle materials taking into account the shape and rigidity of the specimen, and stamps and indenters of different curvatures in the Brazilian test method. The derived equations can be used also for determination of the modulus of elasticity or total deformation modulus and the creep parameters of specimens based on the experimental data of contraction of specimen diameter.

The main solutions take into account the nonlinear interaction between the loading bearing platens and specimen, curvatures of the jaw surfaces, and elastic parameters of the materials.

Numerical examples solved using computer program MATLAB are presented (a) to determine the influences of Young’s modulus, Poisson’s ratio and radii of curvatures of specimen and jaws on the loading angle $2\theta_0$ of the nonuniform normal contact stresses, and stress concentration factors $K$ and $C$ acting across the loaded diameter, and (b) to illustrate the possibilities of using the obtained equations to compare with the existing analog formulae derived on the basis of the concentrated or uniformly distributed normal stresses on the contact surface of given width or loading angle $2\theta_0$.

The application of models of concentrated loads and corresponding equation of tensile stresses always gives a greater or smaller overestimation of the maximum tensile stress in the center of disk where the initial crack is generated. Consequently, this causes the overestimation of the tensile strength of a specimen in the Brazilian test. Greater overestimation occurs for low-modulus material especially at the curved platens, and smaller overestimation occurs for rigid, brittle specimen at the sharp indenters.

The models with more realistically distributed elliptical contact stresses always give somewhat less tensile stress in the center of the disc, and therefore, the overestimation of tensile strength of a specimen is less than that obtained from the models with a concentrated load. But this difference depends on the rigidity and strength of the specimen material. It is greater for soft materials with low modulus and high strength, and smaller for rigid and brittle specimens with low strength.

These relationships are demonstrated above in tabular and graphical representations of the stress concentration factor $K$ for Tests 1 and 2.

### Appendix

This appendix includes calculation examples of tensile strength of the specimens by indirect method. Fig. A1 shows the graphical representations of the stress concentration factor $K$ for Tests 1 and 2.

#### Test 1. The specimen of high rigidity in the flat loading platens.

**Parameters of Test 1**

- Radius of cylindrical specimen, $R_1 = 3$ cm
- Radius of loading platens, $R_2 \rightarrow +\infty$
- Thickness of the specimen, $t = 1.5$ cm
- Peak load, $P = 20$ kN
- Young’s modulus of the specimen, $E_1 = 20$ GPa
- Young’s modulus of the platens, $E_2 = 200$ GPa
- Poisson’s ratio of the specimen, $\nu_1 = 0.16$
- Poisson’s ratio of the platens, $\nu_2 = 0.3$

**MATLAB code for Test 1 (Copy-paste as ready M-file)**

```matlab
R1=3; l=1.5; R2=10*10; r=[0.01:0.01:R1]; % cm.
P=20000 % kg.
E1=2*10^6; E2=2*10^6; % kg/cm^2.
nu1=0.16; nu2=0.3;
fmt=["%0.5f cm.

%-----------------------------

a=(4*P*R1^2/(pi*(R1-R2)^2))/((1-nu1^2)/(E1)+(1-nu2^2/(E2)));
fmt=["%0.5f cm.

teta=asin(a/(R1)^180/pi % degree

b=3*R1^3/(20*pi^3);
c=((8*a^2-3*R1^2)/R1^2+2*r^2)/R1^2+R1^4/r^4/4+atan(sqrt(R1^2+...a^2))/2)+atan(sqrt(R1^2+a^2))/2);%
da=(R1^2-2*a^2)/R1^2+R1^4/r^4/4+atan(a/R1)-pi);%
e=(2*R1^2-2*a^2)/R1^2+R1^4/r^4/4+atan(a/R1)-pi);%
k=b*c^2-e);
sigmaM=P*(pi*R1^2)*x^2/cm^2
plot(kr): xlabel ('K'); ylabel ('r(cm)');
grid on

Results

When r = 0.01, 0.02, 0.03, ..., 2.98, 2.99, 3 cm, we have:

$k = -0.9961$, $-0.9971$, $-0.9971$, ..., 27.0644, 30.5285, 34.3318; $ao = 0.16$ cm;

$2\theta = 62.9^\circ$; $\sigma_s(r=0) = KP/(\pi RL) = -0.996*141.47 = -140.1$ kg/cm^2.
```

### Conflict of interest

The author confirms that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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Test 2. The specimen of low rigidity in the curved loading platens. Parameters of Test 2

- Radius of cylindrical specimen, $R_1 = 3$ cm
- Radius of curved loading platens, $R_2 = 4.5$ cm
- Thickness of the specimen, $L = 1.5$ cm
- Peak load, $P = 20$ kN
- Young's modulus of the specimen, $E_1 = 2$ GPa
- Young's modulus of the platens, $E_2 = 200$ GPa
- Poisson's ratio of the specimen, $v_1 = 0.16$
- Poisson's ratio of the platens, $v_2 = 0.3$

MATLAB code for Test 2

The code is the same as that for Test 1, with the replacement of the value of $E_1 = 2*10^4$ and $R_2 = 4.5$.

Results

When $r = 0.01, 0.02, 0.03, ..., 2.98, 2.99, 3$ cm, we have:

$$k = -0.9164, -0.9163, -0.9163, ..., 6.2465, 6.3843, 6.5239;$$
$$a = 0.867 \text{ cm}; 2\pi = 33.5; \sigma_0 = \frac{KP}{(\pi R_1 L)} = -0.916 \times 141.47 = -129.6 \text{ kg/cm}^2.$$

Fig. A1. Graphical representations of the stress concentration factor $K$ for (a) Test 1 and (b) Test 2.

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Levan Japaridze received his Master of Design and Construction of Underground Structures at St.Petersburg Institute of Mining Geomechanics and Surveying (VNIMI) in 1968 and PhD at Moscow Mining Institute in 1978. He worked as Professor of Mining Engineering at the Georgian Technical University since 1972 in Rock Mechanics, Theory and Design of Tunnels and Underground Constructions. Prof. Levan Japaridze is now a member of the Georgian National Academy of Sciences, the Georgian Engineering Academy, and the Russian Academy of Mining Sciences.