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Strength criterion for rocks under compressive-tensile stresses and its application

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A B S T R A C T

Estimating in-situ stress with hydraulic borehole fracturing involves tensile strength of rock. Several strength criteria with three parameters result in tensile strengths with great differences, although they may describe the relation between strength of rock and confining pressure with low misfits. The exponential criterion provides acceptable magnitudes of tensile strengths for granites and over-estimates for other rocks, but the criterion with tension cut-off is applicable to all rocks. The breakdown pressure will be lower than the shut-in pressure during hydraulic borehole fracturing, when the maximum horizontal principal stress is 2 times larger than the minor one; and it is not the peak value in the first cycle, but the point where the slope of pressure-time curve begins to decline.

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1. Introduction

Numerous tests have been carried out to determine the strengths of rocks under confining pressure (CP), as rocks in-situ are usually under compression state. However, tension usually appears in the vicinity of excavation and borehole, and the tensile strength of each rock is much lower than the compressive strength.

The direct tension test is difficult to perform for rock (You et al., 2006). In other hand, the Brazilian splitting test with rock disc is easy to carry out in laboratory and provides a reasonable estimation for the uniaxial tensile strength (UTS), although there are many issues argued all along (Fairhurst, 1964; Hudson et al., 1972; Efimov, 2009; Yu et al., 2009; You et al., 2011).

Many strength criteria have been proposed to describe the state of stresses in rock at failure, as reviewed in Yu (2002) and You (2011). Clearly, an ideal strength criterion needs to closely fit test data with acceptable accuracy over the stress state expected in practice. Therefore, the tensile strength predicted by a strength criterion is usually used in evaluating the criterion (Ghazvinian et al., 2008; Bineshian et al., 2012). Tensile strengths predicted by both the Coulomb criterion and the Griffith criterion are much higher than the measured magnitudes of almost all rocks, although the two criteria have clear physical backgrounds (Jaeger et al., 2007).

Another issue is the effect of compressive stress on tensile strength, i.e. strength criterion in tension-compression region. It has practical utilization in the in-situ stress estimation with hydraulic breakout of borehole and some cases of wellbore stability.

This paper discusses four criteria with three material-dependent parameters using test data of nine rocks from the published literature. The exponential criterion with tension cut-off is recommended and adopted to estimate in-situ stress with hydraulic borehole fracturing.

2. Strength criteria

Coulomb criterion was initially proposed in 1773 for determination of the shear strength of soil, and introduced for rocks later. It is a linear equation with the principal stresses as

$$\frac{\sigma_5}{\sigma_c} = 1 + m \frac{\sigma_3}{\sigma_c}$$  \hspace{1cm} (1)

where $\sigma_5$ is the major principal stress or rock compressive strength, $\sigma_3$ is the minor principal stress, $\sigma_c$ is the uniaxial compressive strength (UCS), and $m$ is a material-dependent parameter. However, test results from cylindrical specimens of rocks compressed under CP of $\sigma_2 = \sigma_3$ exhibit convex curves of strength. Therefore, many nonlinear criteria were proposed as modifications to the Coulomb criterion, and briefly reviewed as follows.

Hobbs (1964) proposed an empirical criterion with three parameters:

$$\frac{\sigma_5 - \sigma_3}{\sigma_c} = 1 + m \left( \frac{\sigma_3}{\sigma_c} \right)^n$$  \hspace{1cm} (2)
The two criteria, Eqs. (2) and (4), are not applicable to negative $\sigma_3$ for power number $n$ is less than 1, and are certainly beyond the consideration for compressive-tensile strength.

The Sheorey criterion (Sheorey et al., 1989) normalized with UCS is given in the following form:

$$\frac{\sigma_s}{\sigma_C} = \left(1 + m \frac{\sigma_3}{\sigma_C}\right)^n$$  \hspace{1cm} (5)

The criterion proposed in Carter et al. (1991) is in the similar form. The derivative of $\sigma_3$ to $\sigma_3$ for Eq. (5), and Eq. (4) as well, will be less than 1 when $\sigma_3$ is large enough. That means the differential stress $\sigma_s - \sigma_3$ will decrease with increasing CP. The phenomenon appears really for Solnhofen limestone (Mogi, 2007), and Indiana limestone (Schwartz, 1964) as well, within the test range of CP, as illustrated by You (2011). It is totally different from the common knowledge.

The most famous criterion in power form is the generalized H–B criterion (Hoek and Brown, 1980) that has been widely used in rock engineering (Eberhardt, 2012). The criterion fails to describe strength of ductile rocks, such as limestone and marble under high CP.

Cohesion and friction in rocks do not act simultaneously at one point, and the cohesion will be replaced by the frictional resistance when crack initiates in the rock under compression (You, 2005a). The intact rock under shearing will yield and lose its cohesion, but cracks do not slide macroscopically to increase the friction to the maximum when CP is high enough. The differential stress $\sigma_s - \sigma_3$, or the maximum shear stress equivalently, has an upper limit in rocks, and is able to be described with a general criterion (You, 2012):

$$\sigma_s - \sigma_3 = Q_ao - (Q_ao - Q_0)f(x)$$  \hspace{1cm} (7)

where $Q_0$ is the UCS; $Q_ao$ is the limitation of differential stress when CP increases up to infinite; $f(x)$ is a monotonically decreasing function, and satisfies $f(0) = 1$, $f(\infty) = 0$, and $f'(0) = -1$; and $x$ can be written as

$$x = (Q_0 - 1)\sigma_3$$  \hspace{1cm} (8)

where $K_0$ is the increasing rate of strength at $\sigma_3 = 0$.

The exponential criterion (You, 2009, 2010a) is a specific case of Eq. (7) at

$$f(x) = \exp(-x)$$  \hspace{1cm} (9)

The fractional form

$$f(x) = 1/(1 + x)$$  \hspace{1cm} (10)

for Eq. (7) is equivalent to the criterion in Rafai (2011) and Bineshian et al. (2012), and the later manifested that the criterion was originally proposed in Bineshian (2000). In this paper, we called it as the fractional criterion, which is just parallel to the exponential criterion mentioned above.

The average principal stress $\sigma_m = (\sigma_1 + \sigma_2)/2$ and the maximum shear stress $\tau_m = (\sigma_1 - \sigma_2)/2$ are usually used to construct implicit strength criteria. In fact, the abscissa and ordinate will become $2\sigma_m$ and $\sqrt{2}\tau_m$, respectively, after the coordinates of the principal stresses with the same scale are rotated 45° counterclockwise. Therefore, the implicit criteria are not discussed here.

3. Fitting solutions of strength criteria and tensile strengths predicted

In strength criteria, there are always material-dependent parameters, which are determined by fitting the criteria to test data. Test data of nine rocks, presented in Table 1, including granite, limestone, marble, sandstone, and halite, are cited from the literature (Von Kármán, 1911; Schwartz, 1964; Carter et al., 1991; Haimson and Chang, 2000; Sriapai, 2010; You, 2010a) to evaluate the strength criteria. Average magnitude of strengths with the same CP is used as one datum.

Different solutions of fitting the criteria to test data will be obtained using various statistical methods. The least square method is mostly used for the convenience in mathematical calculation, but the fitting solution will depart significantly from normal data to reduce the squares deviation of abnormal data with huge error. Linear regression for a transformed equation of the H–B criterion may result in an imaginary number of UCS (You, 2010a, 2012). Therefore, the fitting solution on the least absolute deviation, i.e. the least mean misfit, is chosen in this paper.

The average values of the mean misfits for nine rocks are 2.9 MPa, 2.9 MPa, 3.1 MPa, and 3.5 MPa using the Sheorey criterion, the fractional criterion, the exponential criterion, and the generalized H–B criterion, respectively. Each criterion provides the least mean misfits for some rocks. The exponential criterion is the best one for three rocks.

Certainly, the misfit is not the single standard to evaluate strength criteria. As illustrated in You (2010a, 2012), the exponential criterion may expose a few abnormal data of Mizuho trachyte and jinping sandstone with huge misfit. A new example of Maha Sarakham halite (Sriapai, 2010) is shown in Figs. 1 and 2. Clearly, the misfit of the exponential criterion is mainly pronounced for two data indicated with A and B, as shown in Fig. 1. At least, datum A may be pointed as an abnormal one. If the datum is deleted, then

<table>
<thead>
<tr>
<th>Rock type</th>
<th>Number of test data</th>
<th>CP (MPa)</th>
<th>UCS (MPa)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Westerly granite</td>
<td>7</td>
<td>100</td>
<td>201</td>
<td>Haimson and Chang (2000)</td>
</tr>
<tr>
<td>Bonnet granite</td>
<td>13</td>
<td>40</td>
<td>226</td>
<td>Carter et al. (1991)</td>
</tr>
<tr>
<td>Tyndall limestone</td>
<td>9</td>
<td>40</td>
<td>250</td>
<td>Carter et al. (1991)</td>
</tr>
<tr>
<td>Indiana limestone</td>
<td>11</td>
<td>69</td>
<td>45</td>
<td>Schwartz (1964)</td>
</tr>
<tr>
<td>Carrara marble</td>
<td>6</td>
<td>162</td>
<td>137</td>
<td>Von Kármán (1911)</td>
</tr>
<tr>
<td>Georgia marble</td>
<td>10</td>
<td>69</td>
<td>30.6</td>
<td>Schwartz (1964)</td>
</tr>
<tr>
<td>Pottsville sandstone</td>
<td>10</td>
<td>62</td>
<td>62</td>
<td>Schwartz (1964)</td>
</tr>
<tr>
<td>Zhoagu sandstone</td>
<td>10</td>
<td>45</td>
<td>132.4</td>
<td>You (2010a)</td>
</tr>
<tr>
<td>Maha Sarakham halite</td>
<td>9</td>
<td>28</td>
<td>23</td>
<td>Sriapai (2010)</td>
</tr>
</tbody>
</table>
the fitting solution using the exponential criterion is almost the same, but the mean misfit decreases from 1.3 MPa to 0.9 MPa.

The fractional criterion, bold line in Fig. 2, also presents a low misfit for the halite, but misfit is distributed in four data, indicated with A, B, C and D, located in two sides of the fitting solution. If the datum A is deleted, then the fitting solution, thin line in Fig. 2, has a significant change, but the mean misfit only decreases from 1.5 MPa to 1.4 MPa. Compared to the fitting solution using the exponential criterion as shown in Fig. 1, the fractional criterion seems not to well describe the strengths under high CPs; hence, the limitation of differential stress from the fractional criterion, \( Q_n \) presented in Fig. 2, has lower confidence than that from the exponential criterion.

The UTSs predicted by the fitting solutions of four criteria are presented in Tables 2 and 3. The measured tensile strengths of eight rocks with Brazilian test are also presented. The tensile strengths of Westerly granite and Carrara marble are cited from Cai (2010) and Ramsey and Chester (2004), respectively.

The Sheorey criterion and the generalized H–B criterion predict reasonable tensile strengths only for Tyndall limestone and Indiana limestone, respectively. Except for a few rocks, the tensile strengths predicted by the two criteria are lower than the measured magnitudes. The ratios of UCS to UTS predicted by the generalized H–B criterion are 10.6–63 with an average of 31.8 as presented in Table 3, which is much higher than the practical value.

Both Rafai (2011) and Bineshian et al. (2012) extended envelopes of the fractional criterion to tension range. The later argued that the criterion described tensile strengths well, but only test results of coal were presented. As listed in Table 2, however, the fractional criterion presents reasonable UTS values only for Westerly granite and Zhaogu sandstone.

The exponential criterion provides reasonable UTS for four rocks: Bonnet granite, Westerly granite, Zhaogu sandstone, and Georgia marble. The tensile strength predicted by the exponential criterion is 6% lower than the measured magnitude for Bonnet granite, and higher than that of the other rocks, as presented in Table 2. It may be concluded that the exponential criterion tends to overestimate tensile strengths for rocks.

4. Strength criterion for rocks under compressive-tensile stresses

The four criteria have nearly the same average magnitudes of mean misfit for nine rocks. However, UTSs predicted by the four criteria have great difference, as presented in Table 2. Clearly, the magnitude \( T \) of UTS is an independent parameter for rock to be
determined from test; and it is impossible to describe strength for all rocks under compressive-tensile stresses by simply extending a criterion that is optimized from compression test data.

As the measured tensile strength of Bonnet granite, the merely one case, is slightly larger than the predicted value obtained from the exponential criterion, we may consider that the entire envelope of the exponential criterion describes the strength of Bonnet granite under compression-tension on the safety side, as shown in Fig. 3. For other rocks, e.g. Zhaogu sandstone as shown in Fig. 4, the exponential criterion with a tension cut-off at \( \sigma_3 = -T \) is able to describe strength of rocks under compressive-tensile stresses. The exponential criterion is applicable low to \( \sigma_E \) for the major principal stress as shown in Fig. 4.

Tensile strengths predicted by the Sheorey criterion and the generalized H–B criterion, however, are much lower than the measured magnitudes for many rocks, such as Bonnet granite, Zhaogu sandstone, and Georgia marble, as presented in Table 2, and so does the fractional criterion for Georgia marble and Bonnet granite. Therefore, the tension cut-off is not applicable to the three criteria mentioned above.

It should be noted that the strengths of rock especially under low CPs are always dispersed, e.g. six UCSs of Zhaogu sandstone are from 119.7 MPa to 142.9 MPa, as shown in Fig. 4. Therefore, improvement for the tension cut-off as illustrated in You (2012) may not be necessary.

The envelope of the exponential criterion for rocks under compressive-tensile stresses is illustrated nearly as a straight line for the variable \( x \) in the range of \(-T \leq \sigma_3 \leq 0\) for all rocks:

\[
\sigma_1 = Q_0 + K_0 \sigma_3
\]

where

\[
x = \left| \frac{(K_0 - 1)\sigma_3}{Q_0 - Q_0} \right| \ll 1
\]

in the range of \(-T \leq \sigma_3 \leq 0\) for all rocks. Therefore, we may use Eq. (11) instead of the exponential criterion in the tension range to simplify the calculation. The equation is not the Coulomb criterion, but the tangent line of the exponential criterion at \( \sigma_3 = 0 \). The parameter \( K_0 \) is always larger than that in the Coulomb criterion. Tension cut-off is at \( \sigma_E = Q_0 - K_0 T \), and tensile strength under compressive stress in the range of \( \sigma_E \leq \sigma_1 \leq Q_0 \) is

\[
\sigma_T = \frac{(Q_0 - \sigma_1)}{K_0}
\]

Strength criteria mentioned above do not consider the effect of the intermediate principal stress. In fact, the commonly utilized tension test under CP, also known as triaxial extension test, is conducted in a stress state of \( \sigma_3 < 0 < \sigma_2 = \sigma_1 \) (Ramsey and Chester, 2004; You, 2010b), but the extension of conventional triaxial compression criterion to the tensile stress means \( \sigma_3 = \sigma_2 < 0 < \sigma_1 \), for which there is indeed no test result available for rocks.

The biaxial compression strength is not equal to the UTS; therefore, the intermediate principal stress has more or less influence on the strength when the minor principal stress is negative. However, as the tensile strength of rock is much lower than the UCS, the minor principal stress with a negative magnitude, i.e. tensile stress, will not be significantly influenced by the intermediate principal stress at a given major principal stress.

5. Hydraulic fracturing of rocks

Hydraulic fracturing is a borehole field test designed to assess the state of in-situ stress in the earth crust. The overburden stress \( \sigma_3 \) is thought of as acting along the vertical direction, and the others on the horizontal plane named \( \sigma_H \) and \( \sigma_H \), are calculated using three parameters, i.e. breakdown pressure \( P_b \), re-opening pressure \( P_r \), and shut-in pressure \( P_s \) obtained from the pressure-time curve.
The fracture induced hydraulically closes back at shut-in pressure

\[ P_b = \sigma_h \]  \hspace{1cm} (14)

after the pump is shut off. It is strongly recommended that more than one method be used for obtaining the crucial \( P_1 \) (Haimson and Cornet, 2003). The re-opening pressure

\[ P_r = 3\sigma_h - \sigma_H \]  \hspace{1cm} (15)

is also relative to the determining techniques, as it is not the peak value, but the point on the ascending portion of the pressure-time curve in subsequent cycles, where the slope begins to decline from that maintained in the breakdown cycle.

The breakdown pressure \( P_b \) is an identified magnitude taken as the peak pressure attained in the first pressure cycle. In the ISRM suggested methods (Haimson and Cornet, 2003), it can be written as

\[ P_b = 3\sigma_h - \sigma_H + T \]  \hspace{1cm} (16)

which means that the major principal stress \( \sigma_1 \) does not influence the tensile failure of rock. However, it should influence strength criterion in compression-tension region. Based on the elastic fracture mechanics, the far-field tensile stress and inner pressure, i.e. pore pressure \( P_b \), have the same effect on the Griffith crack (You, 2005b), therefore the breakdown pressure is

\[ P_b = 3\sigma_h - \sigma_H + \min \left( \frac{Q_b - \sigma_1}{K_0}, T \right) - P_0 \]  \hspace{1cm} (17)

where \( \sigma_1 \) is lower than the UCS, and parameters \( Q_b \) and \( K_0 \) come from the exponential criterion, as illustrated in Fig. 4 and Eq. (13). This provides another magnitude for \( 3\sigma_h - \sigma_H \), based on the re-opening pressure \( P_b \).

As mentioned above, the breakdown pressure \( P_b \) is the peak pressure in the first pressure cycle. This implies an assumption that \( P_b < P_0 \) or \( P_b < \sigma_h \). Hence, the stresses must satisfy

\[ \sigma_H - 2\sigma_h < \min \left( \frac{Q_b - \sigma_1}{K_0}, T \right) - P_0 \]  \hspace{1cm} (18)

The pore pressure is not lower than the static hydraulic pressure at the test interval, which is 10 MPa at a depth of 1000 m. The tensile strength of rock is around and usually lower than 10 MPa. Therefore, the magnitude of right side of Eq. (18) is around zero.

Certainly, the stress magnitudes obtained from hydraulic fracturing always satisfy Eq. (18). Stresses from five boreholes (Chen et al., 2004; Tan et al., 2004; Kang et al., 2007) are in the region of \( \sigma_H < 2\sigma_h \), as presented in Table 4.

### Table 4

<table>
<thead>
<tr>
<th>Location</th>
<th>Depth (m)</th>
<th>( \sigma_v ) (MPa)</th>
<th>( \sigma_h ) (MPa)</th>
<th>( \sigma_H ) (MPa)</th>
<th>( \sigma_H - 2\sigma_h ) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xinwen</td>
<td>790</td>
<td>20.94</td>
<td>16.56</td>
<td>32.39</td>
<td>-0.73</td>
</tr>
<tr>
<td>Lixian</td>
<td>1220</td>
<td>32.33</td>
<td>22.8</td>
<td>42.1</td>
<td>-3.5</td>
</tr>
<tr>
<td></td>
<td>187</td>
<td>4.95</td>
<td>5.33</td>
<td>8.84</td>
<td>-1.82</td>
</tr>
<tr>
<td></td>
<td>220</td>
<td>5.82</td>
<td>5.16</td>
<td>8.67</td>
<td>-1.65</td>
</tr>
<tr>
<td></td>
<td>33</td>
<td>0.86</td>
<td>4.4</td>
<td>5.4</td>
<td>-3.4</td>
</tr>
<tr>
<td></td>
<td>143</td>
<td>3.78</td>
<td>6.98</td>
<td>12.98</td>
<td>-0.98</td>
</tr>
<tr>
<td>Cipin</td>
<td>154</td>
<td>4.13</td>
<td>6.8</td>
<td>12.05</td>
<td>-1.55</td>
</tr>
<tr>
<td></td>
<td>790</td>
<td>21.17</td>
<td>13.8</td>
<td>21.1</td>
<td>-6.5</td>
</tr>
<tr>
<td>Zigu</td>
<td>393</td>
<td>10.07</td>
<td>11.4</td>
<td>17.88</td>
<td>-4.92</td>
</tr>
<tr>
<td></td>
<td>498</td>
<td>12.75</td>
<td>13.3</td>
<td>20.35</td>
<td>-6.25</td>
</tr>
</tbody>
</table>

However, the real in-situ stresses may be different from Eq. (18). For example, the in-situ stresses measured with overcoring test (Pine et al., 1983; Anderson and Christianson, 2003; Tan et al., 2004; Cai et al., 2010) may exhibit large magnitudes of \( \sigma_H - 2\sigma_h \), as presented in Table 5. Furthermore, magnitudes of \( 3\sigma_h - \sigma_H \) are negative at test locations in the South Crofty and Oskarshamn. The borehole should break under pore pressure about 5 MPa, not need pump pressure at all, if it was poured with water. The borehole of Ertan at a depth of 30 m would crack after drilling, for the magnitudes of \( 3\sigma_h - \sigma_H \) is low to ~9 MPa, about the tensile strength of surrounding rocks.

One principal stress is nearly along the vertical direction in every case presented in Table 5; therefore, the hydraulic fracturing method might be used in these situations, but breakdown pressure \( P_b \) should be lower than the shut-in pressure, i.e. the minor principal stress \( \sigma_h \). In other words, \( P_b \) would not be the peak value in the first cycle, but the point where the slope of pressure-time curve begins to decline.

### 6. Conclusions

None of strength criteria, of which the parameters are determined from conventional triaxial compression strengths, might present reasonable tensile strength for all rocks. Tensile strength of rock is an independent parameter to be determined from test.

The generalized H–B criterion usually presents a tensile strength lower than the measured magnitude. However, the exponential criterion predicts well tensile strengths for granites, but overestimates for other rocks; therefore, the criterion may be extended to tensile-compressive stresses with the tangent line combined a tension cut-off, and applicable to rock engineering, such as the estimation of in-situ stress with hydraulic fracturing.

Breakdown pressure may be lower than the shut-in pressure in some cases; and it is not the peak value in the first cycle, but the point where the slope of pressure-time curve begins to decline.

### Conflict of interest

The author wishes to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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