Analysis of one-dimensional consolidation of soft soils with non-Darcian flow caused by non-Newtonian liquid

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Abstract: Based on non-Darcian flow caused by non-Newtonian liquid, the theory of one-dimensional (1D) consolidation was modified to consider variation in the total vertical stress with depth and time. The finite difference method (FDM) was adopted to obtain numerical solutions for excess pore water pressure and average degree of consolidation. When non-Darcian flow is degenerated into Darcian flow, a comparison between numerical solutions and analytical solutions was made to verify reliability of finite difference solutions. Finally, taking into account the ramp time-dependent loading, consolidation behaviors with non-Darcian flow under various parameters were analyzed. Thus, a comprehensive analysis of 1D consolidation combined with non-Darcian flow caused by non-Newtonian liquid was conducted in this paper.

Key words: one-dimensional (1D) consolidation; non-Darcian flow caused by non-Newtonian liquid; non-uniform distribution of total vertical stress; time-dependent loading

1 Introduction

A linear relationship between flow velocity and hydraulic gradient, i.e. Darcy’s law for flow, has been widely used in the theory of consolidation, which can forecast the settlement rates and dissipation rates of excess pore water pressure. In fact, Darcy’s flow law was founded on permeability test of sandy soil. For fine grained soil, both experiments and field observations indicate that water flow in soft soil may deviate from Darcy’s law under low hydraulic gradients. In 1898, King (1898) reported that, at low hydraulic gradients, the flow velocity in porous media was not proportionally linear to the hydraulic gradient. In addition, many researchers (Hansbo, 1960, 1997, 2003; Swartzendruber, 1962; Miller and Low, 1963; Olsen, 1965, 1985; Elnaggak and Krizek, 1973; Dubin and Moulin, 1985) also observed the deviation of water flow from Darcy’s law by experiments or field observations. Kutilek (1972) summarized existing non-Darcian flow relations, and considered that water flow in soils could be classified into two categories. One was Newtonian liquid, and the other was non-Newtonian liquid. Thus, the deviation of water flow from Darcy’s law may be caused by the non-Newtonian liquid in soils. Elnaggak and Krizek (1973) considered that water in soils may be either free water, Newtonian fluid, or “bond water”. Non-Newtonian fluid and a non-Darcian flow in soils caused by non-Newtonian fluid may exist under low hydraulic gradients. In order to generally describe the non-Darcian flow caused by non-Newtonian fluid, Elnaggak and Krizek (1973) proposed an empirical relationship between flow velocity and hydraulic gradient, which can describe all the experimental data reported. As shown in Fig. 1, this relationship may be governed by

\[ v = k \left[ i - (1 - a)i_1 \left( 1 - \exp \left( -\frac{i_1}{\theta i} \right) \right) \right] \]  

where \( v \) is the flow velocity, \( i \) is the hydraulic gradient, \( k \) is the slope of the asymptote in Fig. 1, \( i_1 \) is the zero-velocity intercept of this asymptote, and \( a \) and \( \theta \) are empirical parameters.
It can be seen from Fig. 1 that permeability increases with increasing hydraulic gradient; \(k\) and \(k_{\min}\) were termed as ultimate permeability and minimum permeability, respectively. \(i_1\) was called by Elnaggak as “apparent threshold gradient”. If \(a\) is equal to one, or \(\theta\) becomes zero, or \(i_1\) becomes zero, the non-Darcian flow described by Eq. (1) will degenerate into Darcy’s law.

Since the permeability increases with increasing hydraulic gradient, water flow in soils may deviate from Darcy’s law during the progress of consolidation, and Eq. (1) can basically describe the relationship between flow velocity and hydraulic gradient. It has a theoretical significance in investigating the influence of this water flow relationship on 1D consolidation behavior of soft soil foundations. In fact, many studies on consolidation behavior with non-Darcian flow were conducted (Teh and Nie, 2002; Xie et al., 2007; Liu et al., 2009; Li et al., 2010), and many interesting results have been obtained. Elnaggak and Krizek (1973) analyzed 1D consolidation with this water flow relationship, but external time-dependent loading in an actual engineering and initial excess pore water pressure changing with depth were not incorporated. According to Zhu and Yin (1998), external loading may cause an increase in total vertical stress which changes with depth and time during the progress of consolidation. Therefore, in this paper, an analysis of 1D consolidation with non-Darcian flow relationship is made, considering a change in total vertical stress in combination with time and depth.

### 2 Assumptions and development of governing equations

#### 2.1 Assumptions

In order to investigate consolidation behavior with non-Darcian flow described by Eq. (1), the following assumptions are made, considering a change in vertical total stress with time and depth:

1. An ideal homogeneous soil layer is saturated.
2. Deformation and water flow only in vertical direction are considered.
3. The relationship between void ratio and effective stress is linear, and the coefficient of compressibility keeps constant during the progress of consolidation.
4. The relationship between flow velocity and hydraulic gradient is governed by Eq. (1), and all the parameters are constant during the progress of consolidation.
5. A casual time-dependent loading is applied to the surface of a soil layer, and vertical additional stress caused by external loading varies linearly with depth.

As shown in Fig. 2, the vertical total stress can be written as

\[
\sigma(z, t) = \begin{cases} 
\sigma_0 & (t \leq t_c) \\
\sigma_0 + \frac{\sigma_1 - \sigma_0}{H} z & (t \geq t_c) 
\end{cases}
\]

where \(H\) is the thickness of a soil layer; \(\sigma(z, t)\) is a
function of total vertical stress with time, $t$, and depth, $z$; $t_c$ is the construction time; $\sigma_0$ is the total vertical stress increase at $z=0$ and $t=t_c$; and $\sigma_1$ is the total vertical stress increase at $z=H$ and $t=t_c$.

It is assumed that the total vertical stress in the clayey layer varies linearly with depth and remains unchanged after time $t_c$, and $t_c=0$ indicates that the external load is applied instantly and the total vertical stress may change with depth. Equation $\sigma_0=\sigma_1$ means that the total vertical stress becomes uniform distribution with depth. Equations $t_c=0$ and $\sigma_0=\sigma_1$ mean that the variation pattern of the total vertical stress is reduced to the basic assumption of Terzaghi’s theory of consolidation.

If external loading is ramp loading, Eq. (2) can be written as

$$
\sigma(z, t) = \begin{cases} 
\sigma_0 + \frac{\sigma_1 - \sigma_0}{H} z \frac{t}{t_c} & (t \leq t_c) \\
\sigma_0 + \frac{\sigma_1 - \sigma_0}{H} z & (t \geq t_c)
\end{cases}
$$

### 2.2 Development of the governing equations

An arbitrary unit cell was taken out from a soil layer. According to aforementioned assumptions, the change in water quantity in the unit cell should be equal to the volumetric change in this unit cell, thus this continuity condition can be written as

$$
\frac{\partial v}{\partial z} = m_v \left( \frac{\partial u}{\partial t} - \frac{\partial \sigma(z, t)}{\partial t} \right)
$$

where $u$ is the excess pore water pressure in the $z$-direction, and $m_v$ is the coefficient of volume constrained compressibility.

By substituting Eq. (1) into continuity condition Eq. (4), the governing equation of consolidation can be obtained by

$$
k \frac{\partial^2 u}{\partial z^2} \left[ 1 - (1-a) \theta \exp \left( -\frac{\theta}{l_i \gamma_w} \frac{\partial \sigma}{\partial z} \right) \right] =
\frac{\partial u}{\partial t} - \frac{\partial \sigma(z, t)}{\partial t}
$$

where $\gamma_w$ is the unit weight of water.

Two commonly encountered boundary conditions of either pervious top and pervious bottom (PTPB), or pervious top and impervious bottom (PTIB), are investigated:

$$
\begin{align*}
&u(0, t) = 0 \\
u(H, t) = 0
\end{align*}
$$

Eq. (6a) means that both sides of the soil layer are freely drained. Eq. (6b) means that one side of the soil layer is freely drained, but the other side is impervious. The initial distribution of excess pore water pressure with depth is $u(z, 0) = \sigma(z, 0)$

Eq. (5) is the governing equation of 1D consolidation with non-Darcy’s law caused by non-Newtonian liquid. If $a$ equals 1, or $\theta$ equals 0, or $l_i$ equals 0, Eq. (5) can be simplified to Terzaghi’s linear equation of 1D consolidation with Darcy’s law. For the complexity of second-order nonlinear partial differential equation, analytical solution for Eq. (5) can hardly be obtained. Thus, finite difference method is adopted in this paper to get the solution for the governing equation.

### 3 Solutions for average degree of consolidation

#### 3.1 Dimensionless form of the governing equations

To simplify the process of solution, the following dimensionless variables are introduced:

$$
Z = \frac{z}{H}
$$

$$
T_c = \frac{kt}{\gamma_w m_v H^2}
$$

$$
T_w = \frac{kt}{\gamma_w m_v H^2}
$$

$$
U = \frac{2u}{\sigma_0 + \sigma_1}
$$

$$
q_h = \frac{\sigma_0 + \sigma_1}{2\gamma_w H}
$$

$$
n_h = \frac{2\sigma_0}{\sigma_0 + \sigma_1}
$$

$$
n_i = \frac{2\sigma_1}{\sigma_0 + \sigma_1}
$$

$$
Q(Z, T_c) = \frac{2\sigma(Z, T_c)}{\sigma_0 + \sigma_1}
$$

In terms of these dimensionless variables, Eqs. (5), (6a), (6b) and (7) can be expressed by
\[
\frac{\partial^2 U}{\partial Z^2} \left( 1 - (1-a) \theta \exp \left( - \frac{q_s \theta \partial U}{i} \right) \right) = \frac{\partial U}{\partial T_v} - \frac{\partial Q}{\partial T_v} 
\]
(9)

\[
U(0, T_v) = 0, \quad U(1, T_v) = 0
\]
(10a)

\[
U(0, T_v) = 0, \quad \frac{\partial U}{\partial Z} \bigg|_{i=1} = 0
\]
(10b)

\[
U(Z, 0) = \frac{2\sigma(Z, 0)}{\sigma_n + \sigma_i}
\]
(11)

3.2 Finite difference solution for excess pore water pressure

To obtain the numerical solutions for average degree of consolidation with non-Darcian flow caused by non-Newtonian liquid, the spatial and time domains are subdivided into difference grids. The spatial domain \(0 \leq Z \leq 1\) is firstly divided into \(n\) parts by equal length \(\Delta Z = 1/n\), and spatial nodal points are denoted as \(Z_i = i\Delta Z\) (\(i = 0, 1, 2, 3, \ldots, n\))

\[
T_{ij} = \sum \Delta T_{ij} (j = 0, 1, 2, 3, \ldots)
\]
(13)

Thus, a difference grid is observed between the spatial domain and the time domain. In terms of Crank-Nicolson difference scheme, Eqs. (8a)–(8h) can be expressed as

\[
\alpha_i^j \alpha U_{i+1}^j - 2(\alpha_i^j \lambda + 1)U_i^j + \alpha_i^j \lambda U_{i-1}^j = -\alpha_i^{j+1} \lambda U_{i+1}^{j+1} + 2(\alpha_i^{j+1} \lambda - 1)U_{i+1}^{j+1} - \alpha_i^{j+1} \lambda U_{i-1}^{j+1} - 2(Q_i^j - Q_{i+1}^j)
\]

\(i = 0, 1, 2, 3, \ldots, n; j = 1, 2, 3, \ldots\)
(14)

where \(U_i^j\) is the dimensionless value of excess pore water pressure at \(Z = Z_i\) and \(T_v = T_{ij}\), \(Q_i^j\) is the dimensionless value of total vertical stress at \(Z = Z_i\), \(T_v = T_{ij}\), and \(\lambda = \Delta T_v / (\Delta Z)^2\). \(\alpha_i^{j+1}\) is decided by

\[
\alpha_i^{j+1} = 1 - (1-a) \theta \exp \left( - \frac{q_s \theta U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2\Delta Z} \right)
\]
(15)

The boundary and initial conditions described by Eqs. (10a)–(10b) and (11) may be rewritten in terms of the discrete nodal points as

\[
\begin{align*}
U_0^j & = 0, \quad (j = 0, 1, 2, 3, \ldots) \\
U_n^j & = 0
\end{align*}
\]
(16a)

\[
\begin{align*}
U_0^j & = 0, \quad (j = 0, 1, 2, 3, \ldots) \\
U_{n+1}^j & = U_n^j
\end{align*}
\]
(16b)

\[
U_i^j = \frac{2\sigma(Z, 0)}{\sigma_n + \sigma_i} (i = 0, 1, 2, 3, \ldots, n)
\]
(17)

In terms of matrix, the Eq. (14) can be expressed considering the boundary conditions and initial conditions as

\[
AU = B
\]
(18)

For a linear parabolic partial differential equation, Crank-Nicolson difference scheme is absolutely convergent. Unfortunately, there is no general theoretical knowledge of stability and convergence criteria for nonlinear parabolic partial differential equation, thus the linearization technique may be employed in some instances to obtain difference solution for nonlinear partial difference equation. For example, considering variable coefficient as the increment of constants for a given time, the exact solution can be obtained by iteration. Therefore, the elements of matrix \(A\) in Eq. (18) can be obtained by replacing the dimensionless value of excess pore water pressure at time \(T_{ij}\) with that at time \(T_{i(j-1)}\). An approximation solution for matrix \(U\) can be obtained by iteration.

A mathematical problem for Eq. (9) will be encountered at the impervious surface where excess pore water pressure may keep constant. To avoid this mathematical difficulty, non-Darcy law can be substituted by Darcy’s law at the impervious surface. Thus, the expression for boundary condition (Eq. (16b)) in terms of matrixes can be written as

\[
A_{\text{a}} = -2(1 + \lambda)
\]
\[
A_{\text{a}(n-i)} = 2\lambda
\]
(19)

\[
B_{a} = 2(\lambda - 1)U_{a}^{j+1} - 2\lambda U_{a}^{j+1} - 2(Q_{a}^{j} - Q_{a}^{j+1})
\]
(20)

Boundary condition (Eq. (16a)) can be expressed as

\[
A_{\text{a}} = 1
\]
\[
A_{\text{a}(n-i)} = 0
\]
(21)

In addition, if total vertical stress keeps homogeneous distribution with depth, the middle surface of a soil layer is impervious surface. On this condition, Darcy’s law should be adopted instead of non-Darcy law at impervious surface.

3.3 Derivation of average degree of consolidation

If the compressibility and permeability of homogeneous foundation keep constant in the process of consolidation, the average degree of consolidation in terms of deformation is equal to that in terms of stress. The definition of average degree of
consolidation is the ratio of the consolidation settlement at any time to the final consolidation settlement of the foundation, i.e.

$$U_i = \frac{S_i}{S_w} \quad (21)$$

The settlement of the soil layer at any time can follow as

$$S_i = m \int_0^H [\sigma(z,t) - u(z,t)] \, dz \quad (22)$$

The final consolidation settlement of the soil layer can be written as

$$S_w = \int_0^H \left[ \sigma_0 + \left( \sigma_l - \sigma_o \right) \frac{x}{H} \right] \, dz \quad (23)$$

Combining Eqs. (2), (22), (23) and (21), the average degree of consolidation can be expressed as

$$U_i = \left\{ \begin{array}{ll}
\frac{Q(0, T_v)}{n_0} - \int_0^1 U(Z, T_v) dZ & (T_v \leq T_w) \\
1 - \int_0^1 U(Z, T_v) dZ & (T_v \geq T_w)
\end{array} \right. \quad (24)$$

For a particular case of ramp loading, the average degree of consolidation can be written as

$$U_i = \left\{ \begin{array}{ll}
\frac{T_v}{T_w} - \int_0^1 U(Z, T_v) dZ & (T_v \leq T_w) \\
1 - \int_0^1 U(Z, T_v) dZ & (T_v \geq T_w)
\end{array} \right. \quad (25)$$

Since the analytical solution of pore water pressure cannot be obtained, numerical integration must be adopted and linear interpolation points may use the points of a finite differential grid. In terms of numerical integration, the average degree of consolidation can be expressed as

$$U_i = \left\{ \begin{array}{ll}
\frac{Q(0, T_v)}{n_0} - \sum_{i=1}^n U_{i+1} + U_i \frac{1}{2} & (T_v \leq T_w) \\
1 - \sum_{i=1}^n U_{i+1} + U_i \frac{1}{2} & (T_v \geq T_w)
\end{array} \right. \quad (26)$$

where $U_i$ is the dimensionless value of excess pore water pressure at $Z = Z_i$.

For ramp loading, Eq. (26) can be written as

$$U_i = \left\{ \begin{array}{ll}
\frac{T_v}{T_w} - \sum_{i=1}^n U_{i+1} + U_i \frac{1}{2} & (T_v \leq T_w) \\
1 - \sum_{i=1}^n U_{i+1} + U_i \frac{1}{2} & (T_v \geq T_w)
\end{array} \right. \quad (27)$$

### 3.4 Verification of the difference programming

If $a$ equals 1, or $\theta$ equals 0, or $i_1$ equals 0, Eq. (1) degenerates into Darcy’s law, and the theory of consolidation with non-Darcian flow turns into the linear theory of consolidation. Taking ramp loading from time-dependent loading as an example, analytical solution for the theory of consolidation with Darcy’s law has been achieved by Zhu and Yin (1998). To verify the reliability of the difference programming, as shown in Table 1, a comparison is made between the results of average degree of consolidation by the FDM and that by the analytical method.

**Table 1** Comparison of results obtained by FDM and analytical method ($\Delta z = 0.01$, $\lambda = 0.1$, $a = 1$, or $\theta = 0$, or $i_1 = 0$).

<table>
<thead>
<tr>
<th>Time factor $T_v$</th>
<th>Average degree of consolidation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Results by FDM</td>
</tr>
<tr>
<td>$T_v = 0.01$, $n_0 = 1.5$, PTHB</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>0.004 4</td>
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<tr>
<td></td>
<td>0.007 9</td>
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<tr>
<td></td>
<td>0.010 6</td>
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<tr>
<td></td>
<td>0.025 7</td>
</tr>
<tr>
<td></td>
<td>0.046 4</td>
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<td>0.083 8</td>
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<td>0.27</td>
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<td>0.49</td>
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<td></td>
<td>0.88</td>
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<td>1.19</td>
</tr>
</tbody>
</table>

A good agreement between the results by the present numerical method and those by the analytical method can be observed from Table 1. The maximum absolute error of average degree of consolidation in Table 1 is less than 0.01%, and corresponding relative error is less than 1.8%. Therefore, the results of the present numerical method are reliable in computing 1D consolidation with Darcy’s law. The difference of theory of consolidation between Darcy’s law and non-Darcy law in the process of numerical calculation is the variant value of $\alpha_i/\alpha$. For the theory of consolidation with Darcy’s law, $\alpha/\alpha = 1$; otherwise, the value of $\alpha_i/\alpha$ is decided by Eq. (15). If the accuracy of expression for $\alpha_i/\alpha$ can be ensured and the method of iteration is adopted in the difference programming, the numerical results of consolidation with non-Darcian
flow should be reliable.

4 Parametric analyses of consolidation behaviors

4.1 Influence of parameters of flow law on consolidation behavior

As shown in Fig. 3, the parameters of water flow law, \( a \), \( \theta \) and \( i_1 \), significantly influence the rate of consolidation. Fig. 3(a) shows that the rate of consolidation increases with the increasing value of \( a \) in the case of either PTIB or PTPB. \( a=1 \) indicates that the non-Darcian flow described by Eq. (1) is degenerated into Darcian flow, and the rate of consolidation comes to the maximum in this case. The influence of \( \theta \) on consolidation behavior is shown as Fig. 3(b), and the rate of consolidation is the fastest in the case of \( \theta=0 \), furthermore, the greater the value of \( \theta \) is, the slower the rate of consolidation under both boundary conditions is. The influence of \( i_1 \) on the rate of consolidation can be seen from Fig. 3(c), and the rate of consolidation decreases with the increasing value of \( i_1 \).

4.2 Influence of ratio of equivalent water head of external load to soil layer thickness on rate of consolidation

Fig. 4 shows that the influence of parameter \( q_h \) on average degree of consolidation. The rate of consolidation increases with increasing value of \( q_h \) under both boundary conditions. According to Eq. (8e), the definition of parameter \( q_h \) is the ratio of equivalent water head of average total vertical stress in a foundation to the thickness of a soil layer. That is, greater value of average total vertical stresses will increase the rate of consolidation, i.e. the thinner the thickness of a soil layer is, the faster the rate of consolidation is. Therefore, the rate of consolidation with non-Darcian flow described by Eq. (1) is dependent on the value of average total vertical stresses. In addition, the required consolidation time of a soil layer is no longer proportional to the square of drainage distance of a soil layer, because the average degree of consolidation varies with the value of \( q_h \), even if the time factor is the same. Such consolidation behavior can be determined by Eq. (5), which was also observed by Elnaggak and Krizek (1973) in the condition that the external load is instantly applied.
4.3 Influence of non-uniform distribution of total vertical stress on rate of consolidation

Fig. 5 displays the influence of non-uniform distribution of total vertical stress on the rate of consolidation. \( n_0 = 0 \) indicates that the distribution of total vertical stress is up-triangle, and the rate of consolidation is the lowest in the case of PTIB. \( n_0 = 2 \) means that the distribution of total vertical stress is down-triangle, and the rate of consolidation is the fastest in the case of PTIB. The rate of consolidation increases with the increasing value of \( n_0 \) in the case of PTIB. However, in Fig. 5, the consolidation curves almost coincide well with each other under various values of \( n_0 \), and non-uniform distribution of total vertical stress has no effect on the rate of consolidation in the case of PTPB. In addition, as shown in Fig. 6, the distribution of total vertical stress strongly influences dissipation of excess pore water pressure in the case of PTPB. Meanwhile, the area between the consolidation curves and ordinates axis is almost identical, and this can explain why the distribution of total vertical stress has no influence on the rate of consolidation in the case of PTPB.

4.4 Influence of rate of external loading on rate of consolidation

The rate of external loading has influences on the rate of consolidation, and the less the construction time is, the faster the rate of consolidation is in the case of either PTIB or PTPB (Fig. 7). This consolidation behavior with non-Darcian flow is the same as that with Darcian flow.

5 Conclusions

Given an analysis of 1D consolidation with non-Darcian flow caused by non-Newtonian liquid, the following conclusions can be drawn considering a change in total vertical stress with time and depth:

1. The parameters of water flow law significantly influence the rate of consolidation. The average degree of consolidation increases with the increasing value of \( \alpha \), and decreases with the increasing values of \( \theta \) and \( i_1 \) under both boundary conditions.

2. The ratio of the equivalent water head of external loading to the thickness of soil layer strongly influences the rate of consolidation. The greater this
ratio is, the faster the rate of consolidation under both boundary conditions is. The similarity relationship that the consolidation time is proportional to the square of drainage distance of soil layer in the classical theory of consolidation is not true.

(3) Non-uniform distribution of total vertical stress greatly influences rate of consolidation in the case of PTIB, however, it has no effect on the rate of consolidation in the case of PTPB. The dissipation of excess pore water pressure is affected by non-uniform distribution of total vertical stress in the case of either PTIB or PTPB.

(4) The slower the loading rate is, the slower the rate of consolidation is.

However, 1D consolidation of layered soil with non-Darcian flow caused by non-Newtonian liquid is not considered in the paper, and further study can be appreciated.

References

